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Limits of the generalised Tomimatsu–Sato gravitational fields

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Abstract. In a previous paper, the author presented a new three-parameter family of exact asymptotically flat stationary axisymmetric vacuum solutions of Einstein's equations which contains the solutions of Kerr and Tomimatsu–Sato (TS) as special cases. In this paper, we consider two interesting special cases of the previous family which must be constructed by a limiting process. These will be interpreted as a 'rotating Curzon metric' and a 'generalised extreme Kerr metric'. In addition, approximate forms for the original metrics will be given for the cases of slow rotation and small deformation.

1. Introduction

In this paper, we shall discuss certain limiting cases of the generalised Tomimatsu–Sato (TS) solutions of Einstein's equations presented in an earlier paper by the author (Cosgrove 1977, to be referred to as I). These solutions represent the asymptotically flat vacuum gravitational fields exterior to finite rotating bodies whose mass and angular momentum multipoles depend on precisely three parameters, m, q and δ. The physical meaning of these parameters is given by

$$\text{mass} = m, \tag{1.1a}$$

$$\text{angular momentum, } J = m^2 q, \tag{1.1b}$$

$$\text{mass quadrupole, } Q = m^3[q^2 + p^2(\delta^2 - 1)/3\delta^2], \tag{1.1c}$$

where $p^2 = 1 - q^2$. The case $\delta = 1$ is the Kerr solution and the cases $\delta = 2, 3, 4$ are the Tomimatsu–Sato solutions (TS 1973). When δ is an integer, the metric coefficients are rational or polynomial functions of certain prolate spheroidal coordinates and some of the many remarkable properties of these polynomials are outlined in § 11 of I. When δ is not an integer, they are transcendental functions which depend on the solutions of two independent ordinary differential equations. These solutions may be computed efficiently, even in the highly curved inner regions of space–time, from rapidly convergent infinite series. When $q = 0$, the solutions reduce to the spheroidally symmetric static solutions studied by Zipoy (1966) and Voorhees (1970).

All the multipoles of the gravitational field are analytic functions of m, q and δ^{-2} for $-\infty < m < \infty$, $-\infty < q < \infty$, $-\infty < \delta^{-2} < \infty$. However, in I, we considered in detail only the parameter ranges, $-1 < q < 1$ and $0 < \delta^{-2} < \infty$, i.e. δ real, because of our choice of coordinates and, of course, only positive mass was considered. Nevertheless, by replacing prolate spheroidal coordinates with oblate spheroidal and making other minor adjustments, it is easy to apply the formulae and methods of I to the ranges,

$|q|>1$ and $\delta^{-2}<0$, i.e. δ pure imaginary. Only certain limiting cases require special attention. In § 4, we shall discuss the case $\delta=\infty$, the 'rotating Curzon metric', which reduces to the static Curzon metric when $q=0$. The other case, discussed in § 5, is the limit $q\to\pm1$ and $\delta\to0$ such that $p\delta\equiv p_1$ is held fixed. This is a two-parameter solution generalising the extreme Kerr solution and reducing to the latter when $p_1=0$. It is well known (TS 1973) that the limit $q\to\pm1$ with δ held fixed always yields the extreme Kerr metric, irrespective of δ. Before presenting these two exact solutions, we shall consider, in § 3, approximate forms for the general metric for the cases of slow rotation, $|q|\ll1$, and small deformation, $|\delta-1|\ll1$.

Many of the notations and methods of this paper are taken from I and so it is recommended that the reader be acquainted with that paper. In particular, the rather lengthy proofs that the metrics actually satisfy Einstein's vacuum field equations, are asymptotically flat and have all singularities confined to a bounded region of space are given in I and will not be repeated here. Numbered equations in I will be denoted, for example, by I(4.2), to be read as equation (4.2) from I.

2. The generalised Tomimatsu–Sato solutions

Take the metric of (stationary axisymmetric) space–time in the Weyl–Lewis–Papapetrou canonical form,

$$ds^2=e^{2u}(dt-\omega\,d\phi)^2-e^{-2u}[e^{2\gamma}(dr^2+dz^2)+r^2\,d\phi^2], \tag{2.1}$$

where r, z, ϕ and t are cylindrical coordinates and time and u, ω and γ are functions of r and z only. In terms of the metric coefficients, construct the (real) Ernst potential ψ and the complex Ernst potentials, $\mathscr{E}$ and ξ (Ernst 1968, 1974), according to

$$\psi_r=(1/r)\,e^{4u}\omega_z, \qquad \psi_z=-(1/r)\,e^{4u}\omega_r, \tag{2.2}$$

$$\mathscr{E}=e^{2u}+i\psi, \qquad \xi=(1+\mathscr{E})/(1-\mathscr{E}), \tag{2.3}$$

where ψ_r denotes $\partial\psi/\partial r$, etc. The generalised TS solutions are constructed from two ordinary differential equations (DE) of the second order for functions, $H_4(\eta)$ and $K^{(\epsilon)}(\nu,\eta)$, where the new coordinates (ν,η) are defined by

$$\nu=y/x, \qquad \eta=(x^2-1)/(1-y^2), \tag{2.4}$$

where (x, y) are the usual prolate spheroidal coordinates defined by

$$r=\kappa(x^2-1)^{1/2}(1-y^2)^{1/2}, \qquad z=\kappa xy, \qquad \kappa>0. \tag{2.5}$$

The explicit formulae for the metric coefficients and Ernst potentials are:

$$F_1\equiv e^{-2u}=\Lambda K_1^{(\epsilon)}K_1^{(-\epsilon)}, \tag{2.6a}$$

$$F_2\equiv-\psi\,e^{-2u}=\tfrac12K_1^{(\epsilon)}K_2^{(-\epsilon)}+\tfrac12K_2^{(\epsilon)}K_1^{(-\epsilon)}, \tag{2.6b}$$

$$F_3\equiv e^{-2u}(\psi^2+e^{4u})=\Lambda^{-1}K_2^{(\epsilon)}K_2^{(-\epsilon)}, \tag{2.6c}$$

$$\omega=\kappa[2\delta qp^{-1}-2H_2\,e^{-2u}-(1-\nu^2)\psi_\nu\,e^{-4u}], \tag{2.7}$$

$$e^{2\gamma}=(1+1/\eta)^{-\delta^2}\Gamma(\eta), \tag{2.8}$$

where $K_1^{(\epsilon)}$ and $K_2^{(\epsilon)}$, $\epsilon=\pm1$, are the two linearly independent solutions of the linear Fuchsian DE I(3.8) for $K^{(\epsilon)}$ satisfying boundary conditions I(3.9), $K^{(-\epsilon)}$ is related to

$K^{(\epsilon)}$ by I(3.12), and the functions, H_4, H_2, σ_1, σ_2, Γ and Λ, each functions of η only, are related to each other by I(3.3)–I(3.7) and so are all determined by solving either the H_4 equation I(3.1) (or, equivalently, I(10.1)), or the Γ equation, I(10.4) or I(10.5). The parameters, q and δ, appear in the DE for $H_4(\eta)$ and $K^{(\epsilon)}(\nu,\eta)$ and their boundary conditions, and the parameter identifications (1.1) will hold if we take $\kappa = mp\delta^{-1}$ where $p=(1-q^2)^{1/2}$. Where no confusion can arise, the superscript '(ϵ)' will be dropped from $K^{(\epsilon)}$, $K_1^{(\epsilon)}$ and $K_2^{(\epsilon)}$. The second-order K equation may be replaced by the L equation I(10.40) or the third-order F equation I(7.3) which has F_1, F_2 and F_3 as linearly independent solutions.

Knowledge of the function $H_4(\eta)$ is sufficient to determine the metric on the equatorial plane, $\nu=0$ (see I(4.23)), and $e^{2\gamma}$ everywhere. On the infinite redshift surfaces, $\eta=\eta_0, \eta_1, \eta_2, \ldots$, where the η_i are the zeros of $\Gamma(\eta)$, the K equation simplifies (slightly) to a Lamé equation and the metric coefficient $\omega\, e^{2u}$ becomes a rational function of ν. On the symmetry axis, $y=\pm 1$, all of the metric coefficients and Ernst potentials and their normal derivatives assume very simple elementary functional forms. For example, the complex Ernst potential, ξ, and its normal derivative, $\xi_y = \partial\xi/\partial y$ (with x constant), are given on $y=\pm 1$ by

$$\xi = p\frac{(x+1)^\delta + (x-1)^\delta}{(x+1)^\delta - (x-1)^\delta} - iqy, \tag{2.9a}$$

$$\xi_y = \frac{-4i\delta^2 q(x^2-1)^{\delta-1}}{[(x+1)^\delta - (x-1)^\delta]^2}. \tag{2.9b}$$

A ring-shaped curvature singularity resides on the equatorial plane of every second infinite redshift surface ($\eta=\eta_1, \eta_3, \eta_5, \ldots$, but not $\eta=\eta_0$, the outermost surface). These values of η are those zeros of $\Gamma(\eta)$ which are also zeros, as distinct from poles, of $\Lambda(\eta)$. When δ is not an integer, the surfaces, $\eta=0$ (i.e. $x^2=1$) and $\eta=-1$ (i.e. $x^2=y^2$), are curvature singularities. Consequently, the surface $x=1$ is a natural boundary for the exterior vacuum solution. When δ is an integer and $q\neq 0$, these surfaces are non-singular although, if $\delta\neq 1$, the poles, $x^2=1$, $y^2=1$, exhibit a coordinate-type directional singularity (see Economou 1976 and Ernst 1976 for the case $\delta=2$).

3. Slow rotation and small deformation

3.1. *Slow rotation*, $|q| \ll 1$

When $q=0$, the generalised TS solutions reduce to the static Zipoy–Voorhees solutions (Zipoy 1966, Voorhees 1970). For this case

$$H_4(\eta)=\delta^2, \qquad \Gamma(\eta)=1.$$

The most useful form for the perturbation expansion for small q is

$$\Gamma(\eta)\equiv\Gamma(\eta,q^2) = 1+q^2p^{-2}\Gamma_1(\eta)+q^4p^{-4}\Gamma_2(\eta)+q^6p^{-6}\Gamma_3(\eta)+\ldots. \tag{3.1}$$

However, as emphasised in I, this power series in q^2p^{-2} is in no way restricted to small q. It converges extremely rapidly for all q (even $q^2\approx 1$ when η is appropriately rescaled) and all η not near 0 or -1. The function, $\Gamma(\eta)$, as well as the coefficients, $\Gamma_1(\eta)$, $\Gamma_2(\eta)$, ..., are analytic functions of η in the complex η plane cut from $\eta=-1$

to $\eta = 0$. When δ is an integer, the coefficients are polynomials in η^{-1} and the series terminates at $\Gamma_\delta(\eta) \equiv (-1)^\delta \eta^{-\delta^2}$ so that $\Gamma(\eta)$ is a polynomial in η^{-1} of degree δ^2.

To construct the coefficients in (3.1), first construct the hypergeometric function,

$$W = \eta^{-1} {}_2F_1(1+\delta, 1-\delta; 2; -\eta^{-1}), \tag{3.2}$$

satisfying the DE,

$$\eta(1+\eta)W'' + (1+2\eta)W' - \delta^2 \eta^{-1} W = 0. \tag{3.3}$$

Thence construct

$$V = \delta^2 \eta(1+\eta)W'^2 - \delta^4 \eta^{-1} W^2, \tag{3.4}$$

satisfying either of the equivalent DE, I(10.10) or I(10.11). The first two coefficients, $\Gamma_1(\eta)$ and $\Gamma_2(\eta)$, are readily obtainable by quadratures from

$$\Gamma_1' = V, \tag{3.5}$$

$$V\Gamma_2'' - V'\Gamma_2' = V^3 - \delta^8 W^4 W'^2 - \delta^6 \eta^2 (1+\eta)^2 W^2 W'^4$$
$$- 4\delta^8 \eta^{-1}(1+\eta)^{-1} W W' \int_\eta^\infty \lambda(1+\lambda)(2+\lambda)(W(\lambda)W'(\lambda))^2 \, d\lambda. \tag{3.6}$$

Formulae for $\Gamma_3, \Gamma_4, \ldots$ are rather complicated. The coefficient Γ_n is obtained from $\Gamma_1, \ldots, \Gamma_{n-1}$ by substituting the series (3.1) into the third-order equation I(10.4) and selecting the coefficient of $(q^2 p^{-2})^{n+1}$. The resulting third-order linear DE for Γ_n may be integrated completely by three quadratures (see I(10.30)). As a power series in η^{-1}, $\Gamma_n(\eta)$ starts with the term $k_n \eta^{-n^2}$, i.e.

$$\Gamma_n(\eta) = k_n \eta^{-n^2} + O(\eta^{-n^2-1}) \qquad \text{as } \eta \to \infty, \tag{3.7}$$

where k_n is given by I(10.34) and has asymptotic formula I(10.35) for large n. Two most important properties of the series (3.1) are

$$1 - \Gamma_1(\eta) + \Gamma_2(\eta) - \Gamma_3(\eta) + \ldots = (1 + 1/\eta)^{\delta^2}, \tag{3.8}$$

$$\Gamma(\eta, q^2) \equiv (1+1/\eta)^{\delta^2} \Gamma(-1-\eta, 1/q^2). \tag{3.9}$$

The leading terms in the perturbation expansions for H_4, H_2, σ_1, σ_2 and Λ are given by

$$H_4 = \delta^2 + q^2 \eta(1+\eta)V + O(q^4), \tag{3.10a}$$

$$H_2 = -\delta q \eta(1+\eta)W' + O(q^3), \tag{3.10b}$$

$$\sigma_1 = \delta\eta^{-1} - \tfrac{1}{2}\delta^{-1} q^2[\eta(1+\eta)V' + \eta V] + O(q^4), \tag{3.10c}$$

$$\sigma_2 = \delta^2 q W + O(q^3), \tag{3.10d}$$

$$\Lambda = [(1+\eta)^{1/2} + 1]^\delta [(1+\eta)^{1/2} - 1]^{-\delta}(1 + O(q^2)), \tag{3.10e}$$

as $q \to 0$.

To solve the K equation in ascending powers of q, consider, first, the zeroth approximation, i.e. put $q = 0$ in I(3.8). This equation is easily put into hypergeometric form and the particular solutions, K_1 and K_2, are found to be

$$K_1 = [(1+\eta)^{1/2} + 1]^{-\delta}(1-\nu^2)^{-\frac{1}{2}\delta}[(1+\eta)^{1/2} + (1+\eta\nu^2)^{1/2}]^\delta, \tag{3.11a}$$

$$K_2 = \epsilon i[(1+\eta)^{1/2} - 1]^{-\delta}(1-\nu^2)^{-\frac{1}{2}\delta}[(1+\eta)^{1/2} - (1+\eta\nu^2)^{1/2}]^\delta. \tag{3.11b}$$

Higher approximations may now be obtained by quadratures by well known methods. However, we shall derive the first approximation to the complex Ernst potential ξ by a more direct and elegant method. The result will be seen to agree with 'rule (g)' of Tomimatsu and Sato (1973).

In § 7 of I, it was shown that the functions F_1, F_2 and F_3 defined by (2.3) may be obtained as linearly independent solutions of a third-order linear DE and, further, the form of this DE for arbitrary curvilinear coordinates was given. We shall choose the case of spheroidal coordinates (x, y) where x is the independent variable and y is a constant parameter. Note that H_4, H_2, σ_1 and σ_2, which appear as constants in the DE I(3.8) and I(7.3), are now functions of the independent variable. The required F equation is I(7.15) with Θ and Φ given by I(7.20) and I(7.21) respectively. The boundary conditions at $x=\infty$ are given by I(8.9).

Now consider the slow rotation approximation for the F equation correct to order q. From (3.10), we find

$$F_{xxx}+\left(\frac{2x}{x^2-1}-\frac{2x}{1-y^2}\frac{W''(\eta)}{W'(\eta)}\right)F_{xx}+\left(-\frac{2x^2+2+4\delta^2}{(x^2-1)^2}-\frac{4x^2}{(x^2-1)(1-y^2)}\frac{W''(\eta)}{W'(\eta)}\right)F_x$$
$$+\left(\frac{16\delta^2x}{(x^2-1)^3}+\frac{8\delta^2x}{(x^2-1)^2(1-y^2)}\frac{W''(\eta)}{W'(\eta)}\right)F+\mathrm{O}(q^2)=0. \quad (3.12)$$

Two solutions are immediately obvious. They are

$$F_1=\mathrm{e}^{-2u}=\left(\frac{x+1}{x-1}\right)^\delta+\mathrm{O}(q^2), \quad (3.13)$$

and $F_3=F_1^{-1}+\mathrm{O}(q^2)$. The third solution is obtainable by quadratures in the form,

$$F_2=-\psi\,\mathrm{e}^{-2u}=\frac{\delta q y}{(1-y^2)^2}\left[\left(\frac{x-1}{x+1}\right)^\delta\int_x^\infty(z^2-1)\left(\frac{z+1}{z-1}\right)^\delta W'(\eta)\,\mathrm{d}z\right.$$
$$\left.-\left(\frac{x+1}{x-1}\right)^\delta\int_x^\infty(z^2-1)\left(\frac{z-1}{z+1}\right)^\delta W'(\eta)\,\mathrm{d}z\right]+\mathrm{O}(q^3), \quad (3.14)$$

where $\eta=(z^2-1)/(1-y^2)$ in the integrands in (3.14). From (3.2),

$$W'(\eta)=-\eta^{-2}{}_2F_1(1+\delta,1-\delta;1;-\eta^{-1}). \quad (3.15)$$

Now 'rule (g)' of TS states that

$$\xi=\xi_0+\mathrm{i}q\xi_1+\mathrm{O}(q^2),$$

where

$$\xi_0=\frac{(x+1)^\delta+(x-1)^\delta}{(x+1)^\delta-(x-1)^\delta}, \quad (3.16)$$

$$\xi_1=[(x+1)^\delta-(x-1)^\delta]^{-2}\sum_{l=1}^{\infty}a_{2l-1}(x)P_{2l-1}(y), \quad (3.17)$$

where $P_{2l-1}(y)$ is the Legendre polynomial of degree $2l-1$, $P_{2l-1}(1)=1$ and $a_{2l-1}(x)$ satisfies

$$(x^2-1)a''_{2l-1}(x)-(4\delta-2)xa'_{2l-1}(x)+(4\delta^2-2\delta-4l^2+2l)a_{2l-1}(x)=0. \quad (3.18)$$

The result (3.13) verifies (3.16) which is the Ernst potential for the Zipoy–Voorhees metric. From (3.14) and (3.15), the coefficients $a_{2l-1}(x)$ in (3.17) may be calculated explicitly. After some manipulation of infinite series and application of well known theorems on Legendre, ${}_2F_1$ and ${}_3F_2$ functions, the result is

$$a_{2l-1}(x) = 2\delta(-1)^l \frac{\Gamma(\delta+l)\Gamma(l-\frac{1}{2})}{\Gamma(\delta-l+1)\Gamma(2l-\frac{1}{2})l!} x^{2\delta-2l} {}_2F_1(l-\delta, l-\delta+\tfrac{1}{2}; 2l+\tfrac{1}{2}; x^{-2}). \tag{3.19}$$

(Here, Γ is the familiar gamma function of Euler.) This $a_{2l-1}(x)$ satisfies (3.18) thereby verifying TS 'rule (*g*)'. Of course, (3.19) makes the rule more precise by providing the exact numerical coefficient and also is not restricted to integer values of δ.

If δ is an integer, the series (3.17) terminates at $l=\delta$ since for $l \geqslant \delta+1$, a negative integer factorial appears in the denominator of (3.19). In addition, the hypergeometric series in (3.19) terminates so that $a_{2l-1}(x)$ is a polynomial in x^2 of degree $\delta - l$.

3.2. *Small deformation,* $|\delta-1| \ll 1$

The case $\delta=1$ is precisely the Kerr solution. Let us consider, briefly, the approximate form of the metric when

$$\delta^2 = 1+\theta,$$

where θ is small. The calculations are easier in this case than in the case of q small.

Let

$$p^2\Gamma(\eta) = \gamma_0(\eta) + \theta\gamma_1(\eta) + \theta^2\gamma_2(\eta) + \ldots. \tag{3.20}$$

The leading coefficient $\gamma_0(\eta)$ may be obtained from (2.8) using the known functional form of $e^{2\gamma}$ for the Kerr solution. Thus

$$\gamma_0(\eta) = p^2 - q^2\eta^{-1}. \tag{3.21}$$

Substituting the series (3.20) into the third-order Γ equation I(10.4), we find

$$\eta^2(1+\eta)^2\gamma_1''' + \eta(4+5\eta)\gamma_1'' + 2(1+2\eta)\gamma_1' = q^2\eta^{-2},$$

which may be integrated immediately by three quadratures to give

$$\gamma_1 = -q^2(1+\eta^{-1})\ln(1+\eta^{-1}). \tag{3.22}$$

The remaining coefficients, $\gamma_2, \gamma_3, \ldots$, satisfy inhomogeneous DE whose homogeneous parts are the same as for γ_1 and may therefore be solved in terms of elementary functions and quadratures. For example,

$$\gamma_2 = \tfrac{1}{4}q^2p^{-2}\Big((-1+2q^2)(1+\eta^{-1})\ln^2(1+\eta^{-1}) + \eta^{-1}\ln(1+\eta^{-1}) - 2\eta^{-1}\int_\eta^\infty \lambda^{-1}\ln(1+\lambda^{-1})\,d\lambda\Big). \tag{3.23}$$

Alternatively, the rapidly converging series expansion (3.1) may be used to construct a perturbation expansion for $\Gamma(\eta)$ for small θ. If each $\Gamma_n(\eta)$ is expressed as a power series in θ, then $\Gamma_n(\eta)$ starts with the term in θ^{2n-2}, $n \geqslant 1$.

Construction of the perturbation expansion for the full Ernst potentials is rather tedious. It is straightforward in principle to solve the K equation I(3.8). For the zeroth approximation, put $\delta=1$ in I(3.8). On changing dependent variable to

$$M=(1-\nu^2)^{1/2}(1+\eta\nu^2)^{-1/2}K,$$

this DE simplifies to

$$M_{\nu\nu}+\left(\frac{3\eta\nu}{1+\eta\nu^2}-\frac{p\eta}{p\eta\nu+\epsilon iq}\right)M_\nu=0, \tag{3.24}$$

which is readily solved to give the particular solutions,

$$K_1=[p(1+\eta)^{1/2}+1]^{-1}(1-\nu^2)^{-1/2}[(1+\eta)^{1/2}(p-\epsilon iq\nu)+(1+\eta\nu^2)^{1/2}], \tag{3.25a}$$

$$K_2=\epsilon i[p(1+\eta)^{1/2}-1]^{-1}(1-\nu^2)^{-1/2}[(1+\eta)^{1/2}(p-\epsilon iq\nu)-(1+\eta\nu^2)^{1/2}], \tag{3.25b}$$

satisfying the boundary conditions I(3.9). Further terms in the perturbation expansions for K_1 and K_2 may be found by solving inhomogeneous DE whose homogeneous parts are the same as for M in (3.24).

It is, however, a little quicker to solve the F equation I(7.15) in spheroidal coordinates. The results, to the first approximation, are:

$$\begin{aligned}(x^2-1)(\gamma_0+\theta\gamma_1+\theta^2\gamma_2+\ldots)\,e^{-2u}\\ =(px+1)^2+q^2y^2+\theta p^{-1}\Big(-q^2(px^2-py^2+x)\ln\frac{x^2-y^2}{x^2-1}\\ +\tfrac{1}{2}[(px+1)^2+q^2y^2]\ln\frac{x+1}{x-1}-q^2y\ln\frac{x+y}{x-y}\Big)+O(\theta^2),\end{aligned} \tag{3.26}$$

$$(x^2-1)(\gamma_0+\theta\gamma_1+\theta^2\gamma_2+\ldots)\psi\, e^{-2u}=-2qy-\theta q\Big(y\ln\frac{x^2-y^2}{x^2-1}+x\ln\frac{x+y}{x-y}\Big)+O(\theta^2). \tag{3.27}$$

The complex Ernst potential ξ is given by

$$\begin{aligned}\xi=px-iqy+\theta p^{-1}\Big(-\tfrac{1}{2}q(qx+ipy)\ln\frac{x^2-y^2}{x^2-1}\\ +\tfrac{1}{4}[1-(px-iqy)^2]\ln\frac{x+1}{x-1}-\tfrac{1}{2}iq(px-iqy)\ln\frac{x+y}{x-y}\Big)+O(\theta^2).\end{aligned} \tag{3.28}$$

As for the unperturbed Kerr solution, this space–time has two infinite redshift surfaces. They are the ellipsoid-like surfaces,

$$p^2x^2=1-q^2y^2-2\theta q^2p^{-2}\ln|q|\,.\,(1-y^2)+O(\theta^2).$$

However, the inner surface, which carries the equatorial ring singularity, is contained within the singular surface $x=1$. The outer surface is non-singular except at the poles, $y^2=1$, where it touches the surface $x=1$. 4Note that, in these formulae, the coordinates (x, y) and (ν, η) also depend on θ because $\kappa=mp\delta^{-1}=mp(1-\tfrac{1}{2}\theta+\ldots)$ and a similar comment applies to the case of q small.

4. The rotating Curzon metric, $\delta=\infty$

The limit $\delta\to\infty$ is quite regular if viewed in canonical (r, z) coordinates. This is well known in the static case, $q=0$. The result is the Curzon metric (Curzon 1924, Voorhees 1970),

$$u=-m\rho^{-1},\qquad \omega=0,\qquad \gamma=-\tfrac{1}{2}m^2\rho^{-2}\sin^2\theta,\qquad \xi=\coth(m\rho^{-1}),\tag{4.1}$$

where the coordinates (ρ,θ) are defined by $r=\rho\sin\theta$, $z=\rho\cos\theta$. The rotating Curzon metric, which we are about to construct, has mass m, angular momentum m^2q and mass quadrupole $\frac{1}{3}m^3(1+2q^2)$.

Since $\kappa\to 0$ as $\delta\to\infty$, the coordinates (ν,η) and (x,y) are not defined when $\delta=\infty$. This difficulty is easily avoided if we choose rescaled coordinates (s,λ) as follows:

$$s=\delta\nu,\qquad \lambda=\delta^{-2}\eta.\tag{4.2}$$

The limiting forms as $\delta\to\infty$ are

$$s=mp\rho^{-1}\cos\theta,\qquad \lambda=(mp)^{-2}\rho^2\operatorname{cosec}^2\theta.\tag{4.3}$$

The theory of the generalised TS solutions may be readily transferred to the $\delta=\infty$ case by rescaling, appropriately, the functions, H_4, H_2, etc, before taking the limit. Rescaled quantities will be denoted by a tilde. Inspection of equations I(6.4)–I(6.7), which express H_4, H_2, σ_1 and σ_2 in terms of the metric coefficients, u and ω, reveals that we should rescale H_4, H_2, σ_1, σ_2, Γ and Λ as follows:

$$\tilde H_4=\delta^{-2}H_4,\qquad \tilde H_2=\delta^{-1}H_2,\qquad \tilde\sigma_1=\delta\sigma_1,\tag{4.4a, b, c}$$

$$\tilde\sigma_2=\sigma_2,\qquad \tilde\Gamma=\Gamma,\qquad \tilde\Lambda=\Lambda.\tag{4.4d, e, f}$$

Interpret $\tilde H_4$ as $\tilde H_4(\lambda)$, etc. Hereafter, the tilde will be omitted from $\tilde\Gamma$ and $\tilde\Lambda$. It is straightforward, now, to rewrite the defining relations, I(3.3)–I(3.7), for $\delta=\infty$.

With a prime denoting $\mathrm{d}/\mathrm{d}\lambda$, the two equivalent equations, I(3.1) and I(10.1), for H_4 become

$$\lambda^4\tilde H_4''^2=4\tilde H_4'(-\tilde H_4+\lambda\tilde H_4')(-1+\tilde H_4-\lambda\tilde H_4'),\tag{4.5a}$$

$$\lambda^4\tilde H_4'''+2\lambda^3\tilde H_4''+6\lambda^2\tilde H_4'^2-8\lambda\tilde H_4\tilde H_4'+2\tilde H_4^2+4\lambda\tilde H_4'-2\tilde H_4=0,\tag{4.5b}$$

and the boundary condition at $\lambda=\infty$ is

$$\tilde H_4=p^{-2}+O(\lambda^{-1}).\tag{4.6}$$

From (4.5a, b), two equivalent DE for Γ may be constructed.

As before, the most efficient way of solving the Γ equation is to construct the power series

$$\Gamma\equiv\Gamma(\lambda,q^2)=1+q^2p^{-2}\Gamma_1(\lambda)+q^4p^{-4}\Gamma_2(\lambda)+\ldots.\tag{4.7}$$

First construct

$$\tilde W=\lambda^{-1/2}I_1(2\lambda^{-1/2}),\tag{4.8}$$

where I_1 is the modified Bessel function of order one. This function satisfies

$$\tilde W''+2\lambda^{-1}\tilde W'-\lambda^{-3}\tilde W=0.\tag{4.9}$$

Then construct

$$\tilde V=\lambda^2\tilde W'^2-\lambda^{-1}\tilde W^2\tag{4.10a}$$

$$=\lambda^{-2}{}_1F_2(\tfrac{1}{2};1,2;4\lambda^{-1}),\tag{4.10b}$$

where ${}_1F_2$ is a special case of the generalised hypergeometric function, ${}_pF_q$. $\tilde{V}$ satisfies either of the two equivalent DE,

$$(\lambda^2\tilde{V}''+4\lambda\tilde{V}'+2\tilde{V})^2=4\lambda\tilde{V}'^2+12\tilde{V}\tilde{V}'+8\lambda^{-1}\tilde{V}^2, \tag{4.11a}$$

$$\lambda^4\tilde{V}'''+8\lambda^3\tilde{V}''+(14\lambda^2-4\lambda)\tilde{V}'+(4\lambda-6)\tilde{V}=0. \tag{4.11b}$$

These two functions may be obtained from the W and V of § 3.1 by the rescalings,

$$\tilde{W}=\delta^2 W, \qquad \tilde{V}=\delta^2 V. \tag{4.12}$$

The coefficient Γ_1 and its derivative Γ_1' are given explicitly by

$$\Gamma_1=-2\lambda^3\tilde{W}'^2-\lambda^2\tilde{W}\tilde{W}'+2\tilde{W}^2, \tag{4.13a}$$

$$\Gamma_1'=\tilde{V}, \tag{4.13b}$$

and Γ_2 may be found by quadratures from

$$\tilde{V}\Gamma_2''-\tilde{V}'\Gamma_2'=\tilde{V}^3-\lambda^4\tilde{W}^2\tilde{W}'^4-4\lambda^{-2}\int_\lambda^\infty \mu^3(\tilde{W}(\mu)\tilde{W}'(\mu))^2\,d\mu. \tag{4.14}$$

To compute further coefficients, substitute the series (4.7) into the third-order Γ equation derived from (4.5a). The coefficient of $(q^2p^{-2})^{n+1}$ yields a third-order DE for $\Gamma_n(\lambda)$ of the form

$$(\lambda^4\tilde{V}''+4\lambda^3\tilde{V}'+2\lambda^2\tilde{V})\Gamma_n'''+[4\lambda^3\tilde{V}''+(16\lambda^2-4\lambda)\tilde{V}'+(8\lambda-6)\tilde{V}]\Gamma_n''$$
$$+[2\lambda^2\tilde{V}''+(8\lambda-6)\tilde{V}'+(4-8\lambda^{-1})\tilde{V}]\Gamma_n'=G_n(\lambda), \tag{4.15}$$

where $G_n(\lambda)$ is a quartic polynomial (cubic if $n=2$) in $\Gamma_1, \Gamma_2, \ldots, \Gamma_{n-1}$ and their derivatives up to the third order. An integrating factor for (4.15) is the function, $(\tilde{W}\tilde{W}')^{-2}\tilde{V}$. Hence, a first integral for (4.15) is

$$\tilde{V}\Gamma_n''-\tilde{V}'\Gamma_n'=\tfrac{1}{2}\lambda^{-2}\tilde{W}\tilde{W}'\int_\lambda^\infty (\tilde{W}(\mu)\tilde{W}'(\mu))^{-2}\tilde{V}(\mu)G_n(\mu)\,d\mu. \tag{4.16}$$

It is now straightforward to obtain Γ_n explicitly by two more quadratures.

The series (4.7) converges extremely rapidly. As a power series in λ^{-1}, Γ_n starts with the term $k_n\lambda^{-n^2}$, i.e.

$$\Gamma_n(\lambda)=k_n\lambda^{-n^2}+O(\lambda^{-n^2-1}) \qquad \text{as } \lambda\to\infty, \tag{4.17}$$

where

$$k_n=(-1)^n\{n^n(n^2-1)^{n-1}(n^2-4)^{n-2}(n^2-9)^{n-3}\ldots[n^2-(n-1)^2]\}^{-2}. \tag{4.18}$$

The asymptotic form of k_n for large n is

$$|k_n|\sim(0\cdot806285196\ldots)(4n)^{-2n^2}e^{3n^2}n^{-1/6}. \tag{4.19}$$

Two useful results, corresponding to (3.8) and (3.9), are

$$1-\Gamma_1(\lambda)+\Gamma_2(\lambda)-\Gamma_3(\lambda)+\ldots=e^{1/\lambda}, \tag{4.20}$$

$$\Gamma(\lambda, q^2)\equiv e^{1/\lambda}\Gamma(-\lambda, 1/q^2). \tag{4.21}$$

The function $\Gamma(\lambda)$ is analytic throughout the complex λ plane except for an isolated essential singularity at $\lambda=0$.

The function $K = K^{(\epsilon)}$ requires no rescaling, though now we must interpret K as $K(s, \lambda)$. The differential equation for K is

$$K_{ss} + \left(\frac{\lambda s}{1+\lambda s^2} - \frac{\lambda \tilde{\sigma}_1}{\lambda \tilde{\sigma}_1 s + \epsilon i \tilde{\sigma}_2}\right) K_s + \left(-1 + \frac{\tilde{H}_4 + \epsilon i \lambda \tilde{H}_2 s}{1+\lambda s^2} - \frac{\epsilon i \lambda \tilde{\sigma}_1 \tilde{H}_2}{\lambda \tilde{\sigma}_1 s + \epsilon i \tilde{\sigma}_2}\right) K = 0, \qquad (4.22)$$

and the boundary conditions for K_1 and K_2 at $s = 0$ are

$$K_1 = 1, \qquad K_{1s} = \epsilon i(\lambda^{1/2}\tilde{\sigma}_2 - \tilde{H}_2), \qquad (4.23a)$$

$$K_2 = \epsilon i, \qquad K_{2s} = \lambda^{1/2}\tilde{\sigma}_2 + \tilde{H}_2. \qquad (4.23b)$$

This DE is no simpler than the original DE I(3.8). Note that the two regular singularities in I(3.8) at $\nu = \pm 1$ with exponents, $-\frac{1}{2}\delta$ and $\frac{1}{2}\delta$, have coalesced to form a single irregular singularity at $s = \infty$ in (4.22). Thus (4.22) is of type [2, 1, 1] in the Ince (1927) classification scheme. The 'apparent' singularity with exponents 0 and 2 is still present at $s = -\epsilon i \tilde{\sigma}_2(\lambda \tilde{\sigma}_1)^{-1}$. Without this 'apparent' singularity, (4.22) would be equivalent to a Mathieu equation (Arscott 1964). On the infinite redshift surfaces, $\Gamma(\lambda) = 0$, the K equation actually reduces to a Mathieu equation. The optimum power series solution for (4.22), involving four-term recurrence relations and converging throughout the region, $\rho > 0$, $0 < \theta < \pi$ (rapidly except near $\rho = 0$), is obtained by letting $\delta \to \infty$ in the L equation, I(10.40).

Now, the explicit formulae for the metric and Ernst potentials for the rotating Curzon metric are:

$$F_1 = e^{-2u} = \Lambda K_1^{(\epsilon)} K_1^{(-\epsilon)}, \qquad (4.24a)$$

$$F_2 = -\psi\, e^{-2u} = \tfrac{1}{2} K_1^{(\epsilon)} K_2^{(-\epsilon)} + \tfrac{1}{2} K_2^{(\epsilon)} K_1^{(-\epsilon)}, \qquad (4.24b)$$

$$F_3 = e^{-2u}(\psi^2 + e^{4u}) = \Lambda^{-1} K_2^{(\epsilon)} K_2^{(-\epsilon)}, \qquad (4.24c)$$

$$\omega = 2mq - mp(2\tilde{H}_2\, e^{-2u} + \psi_s\, e^{-4u}), \qquad (4.25)$$

$$e^{2\gamma} = e^{-1/\lambda}\Gamma(\lambda). \qquad (4.26)$$

(Note that the formulae I(3.10), I(3.11), I(3.12), which relate $K^{(\epsilon)}$ to $K^{(-\epsilon)}$, are easily adapted to the case $\delta = \infty$.)

On the equatorial plane, $s = 0$,

$$e^{2u} = \Lambda^{-1}, \qquad \omega = 2mq + 2mp\Lambda(\lambda^{1/2}\tilde{\sigma}_2 - \tilde{H}_2). \qquad (4.27)$$

On the symmetry axis, $\theta = 0$ or π,

$$\xi = p \coth(mp/\rho) - iq \cos\theta \qquad (4.28)$$

For the approximate case of slow rotation, $|q| \ll 1$,

$$\xi = \xi_0 + iq\xi_1 + O(q^2),$$

where

$$\xi_0 = \coth(m/\rho),$$

$$\xi_1 = (1 - e^{-2m/\rho})^{-2} \sum_{l=1}^{\infty} b_l(\rho) P_{2l-1}(\cos\theta),$$

where

$$b_l(\rho) = 2(-1)^l \frac{\Gamma(l-\frac{1}{2})}{\Gamma(2l-\frac{1}{2})\, l!} (m/\rho)^{2l}\, {}_1F_1(2l; 4l; -4m/\rho).$$

5. The generalised extreme Kerr metric

If δ and m are kept fixed and the limit $q^2 \to 1$ taken, then all solutions converge to the extreme Kerr metric involving the single parameter m. This was observed by Tomimatsu and Sato (1973) assuming that px and y are held fixed, which is equivalent to holding (r, z) or (ρ, θ) fixed. Kinnersley and Kelley (1974) considered more fanciful limiting procedures ('distinguished limits') in which the canonical coordinates are not fixed but δ is fixed (since only $\delta = 1, 2, 3, 4$ were available) and found a class of new but unphysical solutions. However, a glance at the quadrupole formula (1.1*c*) suggests that the extreme Kerr metric can be generalised if, while $q^2 \to 1$, we make $\delta \to 0$ in such a way that

$$p_1 = p\delta^{-1} = \kappa m^{-1} \tag{5.1}$$

is kept fixed. The resulting two-parameter family of solutions has mass m, angular momentum $\pm m^2$ and quadrupole $m^3(1-\frac{1}{3}p_1^2)$.

Since κ remains finite in the limit, the coordinates (ν, η) and (x, y) and the functions, H_4, H_2, σ_1, σ_2, Γ and Λ, do not require rescaling. So simply put $\delta = 0$ in the H_4 equations I(3.1), I(10.1) and Γ equations I(10.4), I(10.5). The series solution (3.1) may be carried over to this case by rescaling $\Gamma_n(\eta) = \delta^{2n}\tilde{\Gamma}_n(\eta)$ so that the series takes the form

$$\Gamma(\eta) \equiv \Gamma(\eta, p_1^2) = 1 + p_1^{-2}\tilde{\Gamma}_1(\eta) + p_1^{-4}\tilde{\Gamma}_2(\eta) + \dots. \tag{5.2}$$

The first two coefficients are easily found to be

$$\tilde{\Gamma}_1 = -\ln(1+\eta^{-1}), \tag{5.3}$$

$$\tilde{\Gamma}_2 = \tfrac{1}{2}\ln^2(1+\eta^{-1}) - 2\ln(1+\eta^{-1})\int_\eta^\infty \lambda^{-1}\ln(1+\lambda^{-1})\,\mathrm{d}\lambda + 3\int_\eta^\infty \lambda^{-1}\ln^2(1+\lambda^{-1})\,\mathrm{d}\lambda. \tag{5.4}$$

All the $\tilde{\Gamma}_n$ are expressible in terms of elementary functions and quadratures. To obtain a DE for $\tilde{\Gamma}_n$, it is better to substitute (5.2) into the fourth-order Γ equation, I(10.5) with $\delta = 0$. The result is

$$\eta^2(1+\eta)^2\tilde{\Gamma}_n^{(\mathrm{iv})} + 4\eta(1+\eta)(1+2\eta)\tilde{\Gamma}_n''' + (2+14\eta+14\eta^2)\tilde{\Gamma}_n'' + (2+4\eta)\tilde{\Gamma}_n' = F_n(\eta), \tag{5.5}$$

where $F_n(\eta)$ is a homogeneous quadratic polynomial in $\Gamma_1, \dots, \Gamma_{n-1}$ and their derivatives up to the fourth order. The DE (5.5) is exact. Hence a first integral is

$$\eta(1+\eta)[\eta(1+\eta)\tilde{\Gamma}_n''' + (2+4\eta)\tilde{\Gamma}_n'' + 2\tilde{\Gamma}_n'] = -\int_\eta^\infty F_n(\lambda)\,\mathrm{d}\lambda.$$

This is a somewhat simpler third-order DE for $\tilde{\Gamma}_n$ than that which results from the third-order Γ equation I(10.4). Integrating once more,

$$\eta(1+\eta)\tilde{\Gamma}_n'' + (1+2\eta)\tilde{\Gamma}_n' = \int_\eta^\infty \lambda^{-1}(1+\lambda)^{-1}\int_\lambda^\infty F_n(\mu)\,\mathrm{d}\mu\,\mathrm{d}\lambda. \tag{5.6}$$

The next two quadratures are straightforward.

Some useful properties of the coefficients $\tilde{\Gamma}_n$ are

$$\tilde{\Gamma}_n(\eta) = (-1)^n\tilde{\Gamma}_n(-1-\eta), \tag{5.7}$$

$$\tilde{\Gamma}_n(\eta) = k_n\eta^{-n^2} + O(\eta^{-n^2-1}) \qquad \text{as } \eta \to \infty, \tag{5.8}$$

where

$$k_n = (-1)^n \frac{(1^{n-1}2^{n-2}3^{n-3}4^{n-4}\dots(n-1))^4}{\{n^n(n^2-1)^{n-1}(n^2-4)^{n-2}(n^2-9)^{n-3}\dots[n^2-(n-1)^2]\}^2}. \tag{5.9}$$

For large n,

$$|k_n| \sim (0{\cdot}416028158\dots)2^{-4n^2}(2\pi)^{2n}n^{-1/2}. \tag{5.10}$$

It would appear that the series (5.2) converges more rapidly for large p_1 than for small p_1. In fact, just the reverse is true. If p_1 is small, then η is large and it is appropriate to rescale η as $\eta = (1/p_1^2)\lambda$. From (5.8) we see that for small p_1,

$$p_1^{-2n}\tilde{\Gamma}_n(\eta) = p_1^{2n(n-1)}k_n\lambda^{-n^2}(1+O(p_1^2)), \qquad n \geq 1.$$

For $p_1 = 0$,

$$\Gamma = 1 - \lambda^{-1} \qquad \text{where } \lambda = m^{-2}\rho^2 \operatorname{cosec}^2\theta. \tag{5.11}$$

This is precisely the case of the extreme Kerr metric. A similar argument may be applied to the original series (3.1) showing that convergence is most rapid when q is near 0 or ± 1 for a given value of (r, z).

The K equation I(3.8) changes very little when we put $\delta = 0$. The regular singularities at $\nu = \pm 1$ have exponents 0 and 0. Thus both K_1 and K_2 are logarithmically singular at $\nu = \pm 1$, i.e. on the singular surfaces $x^2 = y^2$ or $\eta = -1$. These surfaces, of course, lie beyond the natural boundary of the vacuum metric, $x = 1$, as viewed from the asymptotically flat outer regions.

The explicit formulae for the metric and Ernst potentials are obtained by putting $\delta = 0$ in (2.6a, b, c), (2.7) and (2.8). The additive constant $2\kappa\delta q p^{-1}$ in (2.7) becomes $2mq$ with $q = \pm 1$ and κ becomes mp_1. Similar comments hold for the formulae I(4.23) for the equatorial plane. On the symmetry axis, $y = \pm 1$,

$$\xi = 2p_1\left(\ln\frac{x+1}{x-1}\right)^{-1} - iqy, \qquad \xi_y = -\frac{4iq}{x^2-1}\left(\ln\frac{x+1}{x-1}\right)^{-2},$$

where $q = \pm 1$.

6. Conclusion

In the foregoing, we discussed four limiting cases of the generalised TS solutions of Einstein's equations introduced in I. Two of these were approximate metrics corresponding to slow rotation and small deformation and in the former case we derived the TS 'rule (g)'. The other two limiting cases were exact solutions which we interpreted as a rotating Curzon metric ($\delta \to \infty$) and as a generalised extreme Kerr metric ($q^2 \to 1$, $\delta \to 0$). In these latter two cases, some simplification of the metric functions occurred but not enough to express them in terms of familiar transcendental functions.

Notice that, as in I, we have restricted the parameter ranges so that $\kappa = mp\delta^{-1}$ is a positive real number (or else $\kappa \to 0$ through positive real values). Thus our spheroidal coordinates are prolate rather than oblate. For example, the parameter q in the rotating Curzon metric may take values $|q| \geq 1$ as well as $-1 < q < 1$. The cases $q = \pm 1$ are precisely extreme Kerr metrics. For $q^2 > 1$, we may relabel coordinates and parameters as follows:

$$(\kappa, p, q, \lambda, s) \to (-i\bar{\kappa}, -i\bar{p}, q, -\bar{\lambda}, -i\bar{s}).$$

Similarly, in the case of the generalised extreme Kerr metric, all the multipoles of the gravitational field are polynomials in p_1^2. So p_1 may take pure imaginary values. The case $p_1 = 0$ is again the extreme Kerr metric. For $p_1^2 < 0$, we may relabel as follows:

$$(\kappa, p_1, x, y, \nu, \eta) \rightarrow (\mathrm{i}\bar{\kappa}, \mathrm{i}\bar{p}_1, -\mathrm{i}\bar{x}, y, \mathrm{i}\bar{\nu}, -\bar{\eta}).$$

When $p_1^2 < 0$, the singular surfaces $x^2 = 1$ and $x^2 = y^2$ are no longer present. All that remains of them is a ring-shaped curvature singularity on the equatorial plane at $\bar{x} = 0$, $y = 0$ which is in addition to the equatorial ring singularity lying on the inner of the two infinite redshift surfaces. The radial coordinate $\bar{x}$ may be continued through to negative values and onto an asymptotically flat negative $\bar{x}$ sheet.

Another interesting contraction of the generalised TS solutions, which will be considered in a separate paper, yields a class of exact asymptotically non-flat solutions recently published by Ernst (1977). These solutions arose from a detailed study of the apparent directional singularity at $x = 1$, $y = 1$ in the TS solutions, particularly $\delta = 2$, by Economou (1976) and Ernst (1976).

References

Arscott F 1964, *Periodic Differential Equations; an Introduction to Mathieu, Lamé and Allied Functions* (Oxford: Pergamon)

Cosgrove C M 1977 *J. Phys. A: Math. Gen.* **10** 1481–524

Curzon H E J 1924 *Proc. London Math. Soc.* **23** 477–80

Economou J E 1976 *J. Math. Phys.* **17** 1095–8

Ernst F J 1968 *Phys. Rev.* **167** 1175–8

—— 1974 *J. ath. Phys.* **15** 1409–12

—— 1976 *J. Math. Phys.* **17** 1091–4

—— 1977 *J. Math. Phys.* **18** 233–4

Ince E L 1927 *Ordinary Differential Equations* (London, New York: Longmans and Green) (Reprinted 1956 (New York: Dover))

Kinnersley W and Kelley E F 1974 *J. Math. Phys.* **15** 2121–6

Tomimatsu A and Sato H 1973 *Prog. Theor. Phys.* **50** 95–110

Voorhees B H 1970 *Phys. Rev.* D **2** 2119–22

Zipoy D 1966 *J. Math. Phys.* **7** 1137–43